Error Polynomial

Note Title 11/30/2010

Let $W(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$
where $x_0 = -1$, $x_n = 1$ and the x_j are equally spaced.

Suppose n is even; $n+1$, the number of points is odd. The spacing is $h = 2/n$. Let $N = n/2$, $2N = n$.

$$W(x) = (x+Nh)(x+(N-1)h)\cdots(x+h)\,x\,(x-h)\cdots(x-Nh).$$

Let $x = rh$, $x_{n-1} < x < x_n$, $x_n = 1 = Nh$, $x_{n-1} = (N-1)h$
Then $N-1 < r < N$

$$W(x) = \underbrace{(r+N)h\,(r+N-1)h\cdots(r+2-N)h}_{n-1 \text{ terms}}\,(x-x_{n-1})(x-x_n)$$

$$|W(x)| < h^{n-1}\underset{\overset{\shortparallel}{n}}{(2N)}(2N-1)\cdots(2)\,|(x-x_{n-1})(x-x_n)|$$

$$= h^{n-1}\,n!\,|(x-x_{n-1})(x-x_n)| = n!\,\frac{2}{n^{n-1}}$$

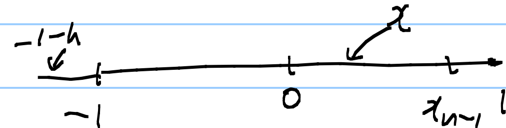
Max of $|(x-x_{n-1})(x-x_n)|$ occurs at $x = \dfrac{x_n+x_{n-1}}{2}$.

Value is $-\left[\dfrac{(x_n+x_{n-1})^2}{4} - \dfrac{(x_n+x_{n-1})^2}{2} + x_n x_{n-1}\right]$

$$= \frac{(x_n-x_{n-1})^2}{4} = \frac{h^2}{4} = \frac{1}{n^2}$$

Hence $|W(x)| \le n!\,\dfrac{2^{n-1}}{n^{n+1}}$, if $x_{n-1} < x < x_n$.

If $0 < x < x_{n-1}$,



$$\left|\frac{W(x+h)}{W(x)}\right| = \left|\frac{(x+(n+1)h)}{(x-nh)}\right| = \frac{|x+1+h|}{|x-1|} > 1$$

since $|x+1+h| > 1$, $|x-1| < 1$

So max value of $|W(x)|$ must be in (x_{n-1}, x_n).

Suppose n is odd. Let $h = \frac{1}{n}$, inteval width $= \frac{2}{n} = 2h$.

$$W(x) = (x+nh)(x+(n-2)h)\cdots(x+h)(x-h)\cdots(x-nh)$$
$$= (x^2-n^2h^2)(x^2-(n-2)^2h^2)\cdots(x^2-h^2)$$

W is even, $W'(x)$ is odd. So $W'(0)=0$.

Let $-h<x<h$. $W'(x) = 2x\cdot\sum$ terms

terms = products of form $(x^2-n^2h^2)\cdots\overbrace{(x^2-k^2h^2)}^{\text{omit}}\cdots(x^2-h^2)$.

Terms $x^2-m^2h^2 < 0$, and always same number.

Hence they combine to give $W'(x) \ne 0$ if $-h<x<h$, $x\ne 0$.

$$|W(0)| = [n(n-2)\cdots 1]^2\cdot h^{n+1} = (n(n-2)\cdots 1)^2\,\frac{1}{n^{n+1}}.$$